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TRIPlE DIRICHLET AVERAGE AND FRACTIONAL DERIVATIVE INVOLVING WITH GENERALIZED K-MITTAG-LEFFLER FUNCTION

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ABSTRACT

The object of this article is to establish some results of Triple Dirichlet averages of Generalized K- Mittag-Leffer functions introduced by Saxena, daiya and Singh [16]. Representations of such relations are obtained in terms of fractional derivative. Some interesting special cases findings.

INTRODUCTION

The Dirichlet average of a function is certain kind of integral average with respect to Dirichlet measure. The concept of Dirichlet average was introduced by Carlson [1,2,4] It is studied, among others, by zu Castell [5], Massopust and Forster [13], Numan [15], Neuman and Van Fleet [16], Daiya [6] and others. A detailed and comprehensive account of various types of Dirichlet averages has been given by Carlson [3] in his monograph..

Deora and Banerji [7] have shown that the Triple Dirlchlet average is equivalent to fractional derivative.

Definition1: Let $k \in R; \alpha, \beta, \gamma \in C; \operatorname{Re}(\alpha) > 0 \; and \; \tau \in C,$ then the generalized k-Mittag-Leffler function is given by Saxena, Daiya and Singh [17]

$$E_{k,\alpha,\beta}^{\gamma,\tau}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^t}{t!} \tag{1}$$

Where $(x)_\tau, (x,\tau \in C)$ denotes the Pochhammer symbol with $(1)_n = n!$ for $n \in N = N \cup \{0\}$, which is defined in terms of gamma function as

$$(x)_\tau = \frac{\Gamma(x+\tau)}{\Gamma(x)} = \begin{cases} 1 & (\tau = 0;\ x \in C \setminus \{0\}) \\ x(x+1)\ldots(x+\tau-1) & (\tau = n \in N;\ x \in C) \end{cases}$$

Special cases of $E_{k,\alpha,\beta}^{\gamma,\tau}(z)$

(i) For τ = q, equation (1) yields generalized K-Mittag-Leffler function defined by Saxena, Daiya and Singh [17]

$$E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^t}{t!} = E_{k,\alpha,\beta}^{\gamma,q}(z) \tag{2}$$

(ii) For k = 1, equation (2) yields generalized Mittag-Leffler function defined by Shukla and Prajapati [19]

$$E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq}}{\Gamma(n\alpha+\beta)} \frac{z^t}{t!} = E_{\alpha,\beta}^{\gamma,q}(z) \tag{3}$$

(ii) When q = 1, equation (2) gives the Mittag-Leffler function defined by Doorego & Cerutti [9]

$$E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,k}}{\Gamma_k(n\alpha+\beta)} \frac{z^t}{t!} = E_{k,\alpha,\beta}^{\gamma}(z) \tag{4}$$

Note 2: A detailed account of Mittag-Leffler function and their application can be found in the survey paper by Haubold et al.[12], Mathai et al. [14], Saxena et al.[18] and Daiya and Ram [6]

I will need some more notations in the further exposition. In the sequel, the symbol E_{n-1} will denote the Euclidean simplex, defined by

$$E_{n-1} = \left[(u_1, \ldots, u_{n-1});\ u_j \geq 0,\ j = 1, 2, \ldots n,\ u_1, \ldots, u_{n-1} \leq 1 \right]. \tag{5}$$

The concept of the Dirichlet average. Following let Ω be a convex set in C and let $z=(z_1,....,z_n)\in\Omega^n; n\geq 2$ and let f be a measurable function on Ω.

Define

$$f(b;z)=\int_{E_{n-1}} f(u\cdot z)\,d\mu_b(u) \tag{6}$$

And

$$(u\cdot z)=\sum_{i=1}^{n-1}u_i z_i+(1-u_1-,...,u_{n-1})z_n \tag{7}$$

and $d\mu_b$ is the Dirichlet measure defined

$$du_b(u)=\frac{1}{B(b)}\,u_1^{b_1-1},....,u_{n-1}^{b_{n-1}-1}\left(1-u_1-...-u_{n-1}\right)^{b_n-1}\,du_1,....,du_{n-1} \tag{8}$$

With the multivariable Beta function

$$B(b)=\frac{\Gamma(b_1),...,\Gamma(b_k)}{\Gamma(b_1+,...,+b_k)} \qquad \mathrm{Re}\left(b_j\right)>0,\ (j=1,2,...,k) \tag{9}$$

For n = 2, we have

$$dm_{\eta,\eta'}(u)=\frac{\Gamma(\eta+\eta')}{\Gamma(\eta)\Gamma(\eta')}\,u^{\eta-1}(1-u)^{\eta'-1}\,du \tag{10}$$

Carlson[2] investigated the average (6) for $f(z)=z^k;\ k\in R$ in the form

$$\operatorname{Re}(b;z)=\int_{E_{n-1}}(u\cdot z)^{k}\,d\mu_b(u) \tag{11}$$

If n = 2 . Carlson[2, 3] proved that

$$R_k(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\int_0^1[ux+(1-u)y]^k u^{\beta-1}(1-u)^{\beta'-1}\,du \tag{12}$$

Where $\beta,\beta'\in C;\min[\operatorname{Re}(\beta),\operatorname{Re}(\beta')]>0\ ;\ x,y\in R$

Let z be species with complex elements Z_{ijk}. let $u=(u_1,\ldots,u_l),v=(v_1,\ldots,v_m)$ and $w=(w_1,\ldots,w_n)$ be an ordered l-tuple, m-tuple and n-tuple of real non-negative weights $\sum u_i=1$, $\sum v_j=1$ and $\sum w_k=1$ respectively (see Deora and Banerji [7])

Define

$$(u\cdot z\cdot v\cdot w)=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}u_i\,z_{ijk}\,v_j w_k \tag{13}$$

If Z_{ijk} is regarded as a point of the complex plane all these convex combinations are points in the convex hull denote by $H(z)$.

Let $u=(u_1,\ldots,u_l)$ be an ordered l-tuple of complex numbers with positive real part $\operatorname{Re}(\mu)>0$ and similarly for $\alpha=(\alpha_1,\ldots,\alpha_m)$ and $\beta=(\beta_1,\ldots,\beta_n)$ then define $dm_\mu(u)$, $dm_\alpha(v)$ and $dm_\beta(w)$ as (13).

Let f be the holomorphic on a domain D in the complex plane if $\operatorname{Re}(\mu)>0,\operatorname{Re}(\alpha)>0,\operatorname{Re}(\beta)>0\ and\ H(z)\subset D$, define

$$F(\mu, z, \alpha, \beta) = \iiint f(u \cdot z \cdot v \cdot w)\, dm_{\mu}(u)\, dm_{\alpha}(v)\, dm_{\beta}(w) \tag{14}$$

Triple Dirichlet average for $(l = m = n = 2)$ of $(u \cdot z \cdot v \cdot w)^t$ is defined by Deora and Banerji [7].

$$\Re_t(\mu, \mu'; z; \alpha, \alpha', \beta, \beta') = \int_0^1 \int_0^1 \int_0^1 (u \cdot z \cdot v \cdot w)^t\, dm_{(\mu,\mu')}(\mu)\, dm_{(\alpha,\alpha')}(v)\, dm_{(\beta,\beta')}(w) \tag{15}$$

where $\mathrm{Re}(\mu) > 0,\ \mathrm{Re}(\mu') > 0,\ \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\alpha') > 0, \mathrm{Re}(\beta) > 0$ *and* $\mathrm{Re}(\beta') > 0,$

and

$$(u \cdot z \cdot v \cdot w) = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \left(u_i \cdot z_{ijk} \cdot v_j \cdot w_k\right)$$

$$= \begin{bmatrix} u_1 z_{111} v_1 w_1 + u_1 z_{112} v_1 w_2 + u_1 z_{121} v_2 w_1 + u_1 z_{122} v_2 w_2 + u_2 z_{211} v_1 w_1 \\ + u_2 z_{212} v_1 w_2 + u_2 z_{221} v_2 w_1 + u_2 z_{222} v_2 w_2 \end{bmatrix} \tag{16}$$

Assume in first species

$$\mathrm{z}_{111} = \mathrm{a},\ \mathrm{z}_{112} = \mathrm{b},\ \mathrm{z}_{121} = \mathrm{c},\ \mathrm{z}_{122} = \mathrm{d}$$

and in second species

$$\mathrm{z}_{211} = \mathrm{e},\ \mathrm{z}_{212} = \mathrm{f},\ \mathrm{z}_{221} = \mathrm{g},\ \mathrm{z}_{222} = \mathrm{h}$$

and $\begin{cases} \mathrm{u}_1 = \mathrm{u} & \mathrm{u}_2 = 1 - \mathrm{u} \\ \mathrm{v}_1 = \mathrm{v} & \mathrm{v}_2 = 1 - \mathrm{v} \\ \mathrm{w}_1 = \mathrm{w} & \mathrm{w}_2 = 1 - \mathrm{w} \end{cases}$

Therefore

$$\left(u\cdot z\cdot v\cdot w\right)=\begin{bmatrix} uvw(a-b-c+d-e+f+g-h)+uv\left(b-d-f+h\right) \\ +vw\left(e-f-g+h\right)+wu(c-d-g+h) \\ +u\left(d-h\right)+v\left(f-h\right)+w(g-h)+h \end{bmatrix} \tag{17}$$

and

$$dm_{\mu,\mu'}\left(u\right)=\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\ u^{\mu-1}\left(1-u\right)^{\mu'-1}\ du \tag{18}$$

$$dm_{\alpha,\alpha'}\left(v\right)=\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\ v^{\alpha-1}\left(1-v\right)^{\alpha'-1}\ dv \tag{19}$$

$$dm_{\beta,\beta'}\left(w\right)=\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\ w^{\beta-1}\left(1-w\right)^{\beta'-1}\ dw \tag{20}$$

Using equation (15)

$$\begin{aligned}\Re_t\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=&\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\int_0^1\int_0^1\int_0^1\big[uvw(a-b-c+d-e+f+g-h)\\ &+uv\left(b-d-f+h\right)+vw\left(e-f-g+h\right)+wu(c-d-g+h)+u\left(d-h\right)+v\left(f-h\right)+w(g-h)+h\big]^t\\ &u^{\mu-1}\left(1-u\right)^{\mu'-1}v^{\alpha-1}\left(1-v\right)^{\alpha'-1}w^{\beta-1}\left(1-w\right)^{\beta'-1}\ du\ dv\,dw.\end{aligned} \tag{21}$$

Let consider the triple average for $(l=m=n=2)$ of $E_{k,\alpha,\beta}^{\gamma,\tau}\left[u\cdot z\cdot v\cdot w\right].$

$$J_t\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k\left(n\alpha+\beta\right)n!}$$

$$\int_0^1\int_0^1\int_0^1\left[uvw(a-b-c+d-e+f+g-h)+uv\left(b-d-f+h\right)+vw\left(e-f-g+h\right)+wu(c-d-g+h)\right.$$
$$\left.+u\left(d-h\right)+v\left(f-h\right)+w(g-h)+h\right]^t u^{\mu-1}\left(1-u\right)^{\mu'-1}v^{\alpha-1}\left(1-v\right)^{\alpha'-1}w^{\beta-1}\left(1-w\right)^{\beta'-1}du\ dv\,dw. \tag{22}$$

FRACTIONAL DERIVATIVE

Fractional derivative with respect to an arbitrary function has bee used by Erdelyi [10]. The general definition for the fractional derivative of order $\alpha\in C_{\infty}$, $\mathrm{Re}(\alpha)>0$ the Riemann-Liouville integral is defined as

$$\left(D_x^{\alpha}F\right)(x)=\frac{1}{\Gamma(-\alpha)}\int_0^x\frac{F(t)}{(x-t)^{1-\alpha}}\,dt \tag{23}$$

and

$$\left(D_{x-x_0}^{\alpha}F\right)(x)=\frac{1}{\Gamma(-\alpha)}\int_{x_0}^{x}\frac{F(t)}{(x-t)^{1-\alpha}}\,dt,\qquad \mathrm{Re}(\alpha)<0 \tag{24}$$

where $F(t)$ is of the form $x^p f(x)$ and $f(x)$ is analytic at $x=0.$

I need Gamma and beta function in the further expression

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \tag{25}$$

We know that

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ (m,n>0) \tag{26}$$

MAIN RESULTS

Theorem 1: Let $k\in R;\ \alpha,\beta,\gamma\in C;\ \mathrm{Re}(\alpha)>0\ and\ \tau\in C,$ then Triple Dirichlet Average is established for $(l=m=n=2)$ of $E_{k,\alpha,\beta}^{\gamma,\tau}\left[u\cdot z\cdot v\cdot w\right].$

$$J_n\left(\mu,\mu',z,\alpha,\alpha',\beta,\beta'\right)=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)}\ \left(y-x\right)^{1-\mu-\mu'}\ D^{-\mu'}\left[E_{k,\alpha,\beta}^{\gamma,\tau}(x)^n(y-x)^{\mu-1}\right] \tag{27}$$

Proof :- using equation (22)

$$J_t\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k\left(n\alpha+\beta\right)n!}$$

$$\int_0^1\int_0^1\int_0^1\left[uvw(a-b-c+d-e+f+g-h)+uv\left(b-d-f+h\right)\right.$$

$$\left.+vw\left(e-f-g+h\right)+wu(c-d-g+h)+u\left(d-h\right)+v\left(f-h\right)+w(g-h)+h\right]^t$$

$$u^{\mu-1}\left(1-u\right)^{\mu'-1}v^{\alpha-1}\left(1-v\right)^{\alpha'-1}w^{\beta-1}\left(1-w\right)^{\beta'-1}\ du\ dv\,dw.$$

For $a=x;e=y;b=c=d=f=g=h=0$ and $t=n$

$$J_n(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}\frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\Gamma(\alpha')}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}$$

$$\int_0^1\int_0^1\int_0^1[uvw(x-y)+vwy]^n\,u^{\mu-1}(1-u)^{\mu'-1}v^{\alpha-1}(1-v)^{\alpha'-1}w^{\beta-1}(1-w)^{\beta'-1}\,du\,dv\,dw.$$

$$J_n(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}\frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\Gamma(\alpha')}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}$$

$$\int_0^1\int_0^1\int_0^1(vw)^n[ux+(1-u)y]^n\,u^{\mu-1}(1-u)^{\mu'-1}v^{\alpha-1}(1-v)^{\alpha'-1}w^{\beta-1}(1-w)^{\beta'-1}\,du\,dv\,dw.$$

Using Beta function, Gamma function and suitable adjustments.

$$J_n(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)n!}$$

$$\int_0^1[ux+(1-u)y]^n\,u^{\mu-1}(1-u)^{\mu'-1}\,du.$$

Using definition of fraction derivative (23), we get

$$J_n(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)}\,(y-x)^{1-\mu-\mu'}\,D^{-\mu'}\left[E_{k,\alpha,\beta}^{\gamma,\tau}(x)^n(y-x)^{\mu-1}\right]$$

This completes the proof of Theorem 1.

Corollary 1.1 Put $\tau = q$ equation (27) reduce in the following from

$$J_n\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{(\alpha)_n\,(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)}\;\left(y-x\right)^{1-\mu-\mu'}\;D^{-\mu'}\left[E_{k,\alpha,\beta}^{\gamma,q}\,(x)^n\,(y-x)^{\mu-1}\right] \tag{28}$$

Corollary 1.2 Put $k=1$ equation (28) reduce in the following from

$$J_n\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{(\alpha)_n\,(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)}\;\left(y-x\right)^{1-\mu-\mu'}\;D^{-\mu'}\left[E_{\alpha,\beta}^{\gamma,q}\,(x)^n\,(y-x)^{\mu-1}\right] \tag{29}$$

Theorem 2: Let $k\in R;\ \alpha,\beta,\gamma\in C;\ \mathrm{Re}(\alpha)>0\ \textit{and}\ \tau\in C,$ then Triple Dirichlet Average is established for $(l=m=n=2)$ of $E_{k,\alpha,\beta}^{\gamma,\tau}\left[u\cdot z\cdot v\cdot w\right]$.

$$J_{-v}\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)\Gamma\left(\alpha+\alpha'\right)\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\mu\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}(1-d)^{1-\mu-\mu'}\,(1-f)^{1-\alpha-\alpha'}\;(1-g)^{1-\beta-\beta'}$$

$$D_{1-d}^{-\mu'}\,D_{1-f}^{-\alpha'}\,D_{1-g}^{-\beta'}\left[E_{k,\alpha,\beta}^{\gamma,\tau}\left(dfg\right)^{-1}\right](1-g)^{\beta-1}\,(1-f)^{\alpha-1}\,(1-d)^{\mu-1} \tag{30}$$

Proof: using equation (22)

$$J_t\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k\,(n\alpha+\beta)\,n!}$$

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$$\int_0^1\int_0^1\int_0^1 [uvw(a-b-c+d-e+f+g-h)+uv(b-d-f+h)$$
$$+vw(e-f-g+h)+wu(c-d-g+h)+u(d-h)+v(f-h)+w(g-h)+h]^t$$
$$u^{\mu-1}(1-u)^{\mu'-1}v^{\alpha-1}(1-v)^{\alpha'-1}w^{\beta-1}(1-w)^{\beta'-1}\,du\,dv\,dw.$$

Set $a=dfg; b=df; c=dg; e=fg; h=1$ and $t=-v$ in (22)

$$J_{-v}(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}\frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\Gamma(\alpha')}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)(-v)!}$$
$$\int_0^1\int_0^1\int_0^1 [uvw(dfg-df-fg-dg+d+f+g-1)+uv(df-f-d+1)+vw(fg-f-g+1)$$
$$+wu(dg-d-g+1)+u(d-1)+v(f-1)+w(g-1)+1]^{-v}$$
$$u^{\mu-1}(1-u)^{\mu'-1}v^{\alpha-1}(1-v)^{\alpha'-1}w^{\beta-1}(1-w)^{\beta'-1}\,du\,dv\,dw.$$

on suitable adjustments of terms,

$$J_{-v}(\mu,\mu',z,\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')}{\Gamma(\mu)\Gamma(\mu')}\frac{\Gamma(\alpha+\alpha')}{\Gamma(\alpha)\Gamma(\alpha')}\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k(n\alpha+\beta)(-v)!}$$
$$\int_0^1\int_0^1\int_0^1 \{[1-u(1-d)][1-v(1-f)][1-w(1-g)]\}^{-v}$$
$$u^{\mu-1}(1-u)^{\mu'-1}v^{\alpha-1}(1-v)^{\alpha'-1}w^{\beta-1}(1-w)^{\beta'-1}\,du\,dv\,dw.$$

For $u=\dfrac{p}{1-d}; v=\dfrac{q}{1-f}; w=\dfrac{r}{1-g}$

$$J_{-\nu}\left(\mu,\mu';z;\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)}{\Gamma\left(\mu\right)\Gamma\left(\mu'\right)}\frac{\Gamma\left(\alpha+\alpha'\right)}{\Gamma\left(\alpha\right)\Gamma\left(\alpha'\right)}\frac{\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\beta\right)\Gamma\left(\beta'\right)}\sum_{n=0}^{\infty}\frac{(\gamma)_{n\tau,k}}{\Gamma_k\left(n\alpha+\beta\right)(-\nu)!}$$
$$(1-d)^{1-\mu-\mu'}(1-f)^{1-\alpha-\alpha'}(1-g)^{1-\beta-\beta'}\int_0^{1-g}\int_0^{1-f}\int_0^{1-d}[(1-p)(1-q)(1-r)]^{-\nu}$$
$$r^{\beta-1}(1-g-r)^{\beta'-1}q^{\alpha-1}(1-f-q)^{\alpha'-1}p^{\mu-1}(1-d-p)^{\mu'-1}\,dp\,dq\,dr.$$

Now using definition of fractional derivatives,

$$=\frac{\Gamma\left(\mu+\mu'\right)\Gamma\left(\alpha+\alpha'\right)\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\mu\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}(1-d)^{1-\mu-\mu'}(1-f)^{1-\alpha-\alpha'}(1-g)^{1-\beta-\beta'}$$
$$D_{1-d}^{-\mu'}D_{1-f}^{-\alpha'}D_{1-g}^{-\beta'}\left[E_{k,\alpha,\beta}^{\gamma,\tau}\left(dfg\right)^{-1}\right](1-g)^{\beta-1}(1-f)^{\alpha-1}(1-d)^{\mu-1}$$

This completes the proof of Theorem 2.

Corollary 2.1 Put $\tau=q$ equation (30) reduce in the following from

$$J_{-\nu}\left(\mu,\mu',z,\alpha,\alpha',\beta,\beta'\right)=\frac{\Gamma\left(\mu+\mu'\right)\Gamma\left(\alpha+\alpha'\right)\Gamma\left(\beta+\beta'\right)}{\Gamma\left(\mu\right)\Gamma\left(\alpha\right)\Gamma\left(\beta\right)}(1-d)^{1-\mu-\mu'}(1-f)^{1-\alpha-\alpha'}(1-g)^{1-\beta-\beta'}$$
$$D_{1-d}^{-\mu'}D_{1-f}^{-\alpha'}D_{1-g}^{-\beta'}\left[E_{k,\alpha,\beta}^{\gamma,q}\left(dfg\right)^{-1}\right](1-g)^{\beta-1}(1-f)^{\alpha-1}(1-d)^{\mu-1} \quad (31)$$

Corollary 2.2 Put $k=1$ equation (31) reduce in the following from

$$J_{-v}(\mu,\mu',z,\alpha,\alpha',\beta,\beta')=\frac{\Gamma(\mu+\mu')\Gamma(\alpha+\alpha')\Gamma(\beta+\beta')}{\Gamma(\mu)\Gamma(\alpha)\Gamma(\beta)}(1-d)^{1-\mu-\mu'}(1-f)^{1-\alpha-\alpha'}(1-g)^{1-\beta-\beta'}$$

$$D_{1-d}^{-\mu'}D_{1-f}^{-\alpha'}D_{1-g}^{-\beta'}\left[E_{\alpha,\beta}^{\gamma,q}(dfg)^{-1}\right](1-g)^{\beta-1}(1-f)^{\alpha-1}(1-d)^{\mu-1} \quad (32)$$

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